APPLICATION OF VARIATIONAL METHODS TO THE INVESTIGATION OF WAVE PROPAGATION IN SYSTEMS WITH TRAVELING PARAMETERS

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Wave propagation in arbitrary linear systems with parameters varying in time and space according to a traveling wave law is considered by using the Lagrangean description. It is shown that the solution of this problem can be reduced to the solution of the stationary problem for a fixed inhomogeneous medium. Consequently, the reflection and transmission coefficients for a moving inhomogeneity can be expressed in terms of the analogous coefficients for the auxiliary fixed layer. Relationships connecting the energy and frequency of the interacting wave packets are obtained from which the equality of the total quantum flux through the surface enclosing the moving inhomogeneous domain to zero follows. Some particular cases are examined.

1. There is a significant number of papers devoted to wave propagation of different nature in systems with parameters varying in time and space in conformity with a traveling wave law (see [1, 2] and the survey [3], for example). A change in the parameters can hence occur either because of the motion of the inhomogeneous medium itself or because of the presence of powerful waves in the system (a sonic or shock wave in a fluid or gas, a pumping wave in nonlinear electrodynamic systems, etc.). However, because of the diversity of the dynamical equations, the results obtained for specific medium models are particular in nature, as a rule; hence, it is interesting to examine the mentioned questions from a single point of view. The possibility hence appears not only of clarifying the degree of generality of systems of different nature and of determining the domain of applicability of any result, but also of investigating new cases and making some general deductions.

The possibility of such a single approach is well known in mechanics since the most general formulation of the motion laws is here given by the principle of least action within the scope of the Lagrangean description of mechanical systems. Since the field equations can also be written in the Lagrangean form for a broad class of distributed systems, it is natural to try to use this apparatus for nonstationary media, of which particular cases are systems with traveling parameters.

The variational principles have been applied to such systems for two limit cases in the literature: slow ("adiabatic") and jump laws of parameter variation. In particular, for systems with slowly varying parameters the approach mentioned permits proving the conservation of the number of quanta in a quasi-monochromatic wave packet independently of the physical nature of the wave [1, 3], and in the case of an abrupt boundary the derivation of some general relationships connecting the total energy and frequency of the interacting wave packets [3, 4] previously established just for individual models of a medium [1, 2].

Meanwhile, the real profile of the moving inhomogeneous domain must be taken into account for a number of practical problems. Even for comparatively thin transition layers their approximation by a jump becomes inapplicable as the frequency of one of the waves rises significantly, as occurs if the wave phase velocity is close to the velocity of boundary motion. The adiabatic approximation also has a limited domain of applicability; for example, it does not permit the analysis of signal reflection for a moving inhomogeneity.

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It is hence interesting to investigate the general case when the system parameters vary arbitrarily according to a traveling wave law. The problem is hence complicated considerably since partial differential equations with variable coefficients must be considered for its solution. Although no exact analytical solution of such a problem has been found successfully in the general case, it turns out to be possible to reduce it to another stationary, auxiliary problem (exactly as had been done in [2] for waves in a moving inhomogeneous plasma) and, consequently, to derive some energy relationships for quasiharmonic wave packets in such systems which will extend the results obtained for sharp boundaries.

2. A broad class of linear systems can be described by a Lagrangean in the form of a quadratic form of generalized coordinates (q) and their space and time derivatives (q_x, q_t, etc.). For brevity, we limit ourselves here to a one-dimensional system with one generalized coordinate described by the density of the Lagrange function

$$L = aq_x^2 + 2bq_xq_t + cq_t^2 + dq^2 \tag{2.1}$$

where a, b, c and d are arbitrary functions of the argument $\zeta = x - Vt$. (The addition of terms of the type q_xq_t and q_tq to the Lagrangean does not influence the subsequent reasoning, and hence the corresponding members in (2.1) are omitted.)

The Lagrange equation for the system (2.1) is

$$aq_{xx} + 2bq_{xt} + cq_{tt} + q_x \frac{\partial}{\partial\zeta}(a - Vb) + q_t \frac{\partial}{\partial\zeta}(b - Vc) - dq = 0 \tag{2.2}$$

Let us go over to new independent variables in (2.2). It is natural to take $\zeta = x - Vt$ as one, and we select the second ($\xi = \xi(x, t)$) from the condition that the coefficient of the mixed derivative $q_{\zeta\xi}$ equals zero. (The coefficient of the first derivative q_ξ hence also vanishes.) As is easy to see, to do this the variable ξ should be an integral of the equation

$$(a - Vb)\,dt = (b - Vc)\,dx$$

from which

$$\xi = t - \int \frac{b - Vc}{\alpha}\,d\zeta \tag{2.3}$$

where $\alpha = a - 2Vb + V^2c$.

After substitution of (2.3), Eq. (2.2) becomes

$$\alpha q_{\zeta\zeta} + \frac{ac - b^2}{\alpha} q_{\xi\xi} + q_\zeta \frac{\partial\alpha}{\partial\zeta} - dq = 0 \tag{2.4}$$

which allows separation of variables. The solution of (2.4) can be sought in the form

$$q = g(\zeta)\exp(i\Omega\xi) \tag{2.5}$$

where $g(\zeta)$ satisfies the equation

$$g_{\zeta\zeta} + g_\zeta \frac{\partial \ln\alpha}{\partial\zeta} - g\left(\frac{d}{\alpha} + \Omega^2 \frac{ac - b^2}{\alpha^2}\right) = 0 \tag{2.6}$$

and Ω^2 is the separation constant.

Let us note that (2.6) retains its form as $V \to 0$. In this case it describes wave propagation in some stationary system ($\zeta' = x$) with the parameters $a(x)$, $b(x)$, $c(x)$, and $d(x)$. Now, if another also stationary system with the parameters

$$a' = \alpha, \quad b' = b, \quad c' = ac/\alpha, \quad d' = d \tag{2.7}$$

is considered, then the equation

$$g_{xx}' + g_x' \frac{\partial}{\partial x}\ln a' - g'\left[\frac{d'}{a'} + \Omega^2 \frac{a'c' - (b')^2}{(a')^2}\right] = 0$$

describing the wave propagation in such a system, will agree with (2.6).

This circumstance permits reduction of the solution of the initial nonstationary problem to the solution of another stationary problem about wave propagation in an auxiliary inhomogeneous system (K'). Its parameters are obtained from the parameters of the initial system (K) by a simple conversion in conformity with (2.7). Such a comparison affords the possibility of using the methods and results of solutions known for stationary systems for nonstationary problems.

3. The solution of the auxiliary (stationary) problem can be written, without limiting the generality, as

$$q' = \sum_l^n Q_l' \exp\left[i\left(\omega' t - \int k_l' dx\right)\right] \tag{3.1}$$

Let a change in the parameters in the system K occur within a finite (moving) domain, which corresponds to a fixed layer in the K' system. Outside the layer, in the domain of constant parameters, Q_l', ω', and k_l' have the sense of the amplitude, frequency, and wave number n of the normal waves existing in such a system. Starting from (2.5), the very same solution q' can be written as follows:

$$q' = g'(\zeta')\exp(i\Omega\xi') = g(x)\exp\left[i\Omega\left(t - \int \frac{b'}{a'}\, dx\right)\right] \tag{3.2}$$

Comparing (3.1) and (3.2) we find

$$\Omega = \omega'$$

$$g(x) = \sum_l^n Q_l' \exp\left[i\left(\omega' \int \frac{b'}{a'}\, dx - \int k_l'\, dx\right)\right]$$

from which there follows for $q(\zeta, \xi)$ in conformity with (2.5):

$$q = \sum_l^n Q_l' \exp(i\theta_l) \tag{3.3}$$

where

$$\vartheta_l = \omega'\left(t + V\int \frac{c}{a'}\, d\zeta\right) - \int k_l'\, d\zeta$$

Defining the frequency and wave numbers in the system K as $\omega_l = \partial\theta_l/\partial t$, $k_l = -\partial\theta_l/\partial t$, we obtain

$$\omega_l = \omega'\left(1 - V^2 \frac{c}{a'}\right) + Vk_l', \qquad k_l = k_l' - V\frac{c}{a'}\,\omega' \tag{3.4}$$

Let us note that the quantities ω_l, k_l found according to (3.4) will satisfy the customary Doppler relationships [3]

$$\omega_0\left(1 - \frac{V}{v_0}\right) = \omega_l\left(1 - \frac{V}{v_l}\right) \tag{3.5}$$

where v_l is the phase velocity in the wave l, and the subscript 0 refers to the incident wave.

As follows from (3.1) and (3.3), the solution of the initial problem outside the domain of parameter variation is also written as the sum of n waves, whose amplitudes Q_l equal the amplitudes Q_l' of the corresponding waves in the system K'. This permits finding the power transformation coefficients T_l of the primary wave into each of the secondary waves if the appropriate coefficients T_l' are known for the auxiliary system K'.

The mean energy flux density in the wave l per period equals [5]

$$\langle s_l\rangle = \left\langle q_t^l \frac{\partial L}{\partial q_x^l}\right\rangle = \langle 2q_t^l(a_l q_x^l + b_l q_t^l)\rangle = Q_l^2\omega_l(a_l k_l - b_l\omega_l) \tag{3.6}$$

where a_l, b_l are Lagrange coefficients in the domain where the wave l is propagated. Hence, expressing the transformation coefficients T_l and T_l' in the systems K and K', we easily obtain

$$T_l = T_l' \frac{\omega_l}{\omega_0}\, \frac{a_l k_l - b_l\omega_l}{a_0 k_0 - b_0\omega_0}\, \frac{a_0' k_0' - b_0'\omega'}{a_l' k_l' - b_l'\omega'} \tag{3.7}$$

The values of T_l' depend primarily on the profile of the auxiliary layer, moreover, for dispersive systems they may depend in a complex manner on ω' and k'. The remaining factors in (3.7) describe effects associated with the nonstationarity of the system K, i.e., with the motion of the inhomogeneity. It is easy to see, for example, that $T_l = 0$ is possible for some relationship of the parameters even if the corresponding coefficient is $T_l \neq 0$. Since the flux $\langle s_l \rangle$ is related to the energy density $\langle w_l \rangle$ by the relationship $\langle s_l' \rangle = \langle u_l w_l \rangle$, where u_l is the group velocity in the wave l, and the amplitude Q_l is finite, this means that $Ku_l = 0$ in the system. The wave l understandably hence entrains energy from the moving inhomogeneous domain as before.

4. Ordinarily only the amplitude and power transformation coefficients are investigated in considering the energy relationships on moving boundaries, as has indeed been done above. Meanwhile, for signals in the form of wave packets it is also important to know the change in their total energy; the transformation of the signal duration (pulse) or its spatial length (Λ) should hence be taken into account. As is easy to show [2, 3], this transformation has the following form for a homogeneous wave packet:

$$\frac{\Lambda_l}{\Lambda_0} = \left| \frac{u_l - V}{u_0 - V} \right| \tag{4.1}$$

Let us examine some energy relations below for quasiharmonic wave packets in systems of the type (2.1). Let us define the number of quanta in a quasiharmonic pulse with energy W_l and frequency ω_l as $N_l = W_l/\omega_l$; the quantity ω_l is evidently algebraic here since the frequency N_l can generally change sign during transformation of a packet, as is easily seen from the Doppler relation (3.5). Let us prove the total quantum flux through some surface enclosing a moving inhomogeneous domain is zero; the corresponding condition can be written as

$$\sum_l^n \{N_l \operatorname{sgn}(u_l - V)\} = \sum_l^n \left\{\frac{W_l}{\omega_l} \operatorname{sgn}(u_l - V)\right\} = 0 \tag{4.2}$$

Here and henceforth the braces will denote the difference in the values of the quantities on different sides of the inhomogeneous domain. Taking account of (4.1) after having substituted $W_l = \langle w_l \rangle \Lambda_l$, we obtain

$$\sum_l^n \left\{\langle \omega_l \rangle \frac{u_l - V}{\omega_l}\right\} = \sum_l^n \left\{\frac{\langle s_l \rangle - V\langle w_l \rangle}{\omega_l}\right\} = 0 \tag{4.3}$$

The mean energy density per period in the wave l equals [5]

$$\langle w_l \rangle = \left\langle q_l^l \frac{\partial L}{\partial q_l^l} \right\rangle = Q_l^2 \omega_l (b_l k_l - c_l \omega_l) \tag{4.4}$$

where it has been taken into account that $\langle L \rangle = 0$ for a harmonic wave. Substituting (3.6), (4.4), into (4.3), we have

$$\sum_l^n \{Q_l^2 [(a_l - Vb_l) k_l - (b_l - Vc_l) \omega_l]\} = 0 \tag{4.5}$$

Expressing ω_l and k_l in (4.5) in terms of ω' and k_l' in conformity with (3.4), we obtain

$$\sum_l^n \{Q_l^2 (\alpha_l k_l' - b_l \omega')\} = 0$$

or taking account of (2.7)

$$\sum_l^n \{Q_l^2 (a_l' k_l' - b_l' \omega')\} = \sum_l^n \{\langle s_l' \rangle\} = 0 \tag{4.6}$$

It is easy to see that condition (4.6) means the total energy flux through a surface enclosing an inhomogeneous domain in the stationary system K' equals zero, as is evident from the conservativeness of this latter. The relationship (4.2) is thereby proved for quasimonochromatic wave packets in an arbitrary Lagrange system of type (2.1) with traveling parameters.

Using (3.5), the relationship (4.2) can also be written as

$$\sum_l^n \left\{ |N_l| \operatorname{sgn}\left[(u_l - V)\left(1 - \frac{V}{v_l}\right)\right]\right\} = 0 \tag{4.7}$$

Depending on the dispersive properties of the system (the signs of $1-V/v_l$) the equality (4.7) can denote both conservation of the total number of quanta in the secondary waves relative to the primary, as well as generation of new quanta because of the energy of the source assuring the motion of the inhomogeneous domain.

Thus, if the difference $1-V/v_l$ has the same sign for all the waves ($l=1$-n), the relationship

$$|N_0| = \sum_{l \neq 0}^{n} |N_l| \tag{4.8}$$

follows from (4.7); i.e., the sum of the quanta in the secondary waves equals the number of quanta in the incident wave. If $1-V/v_l$ is obtained different for one of the secondary waves, then the appropriate member in the right side of (4.8) will enter with a minus; i.e., we obtain

$$|N_0| = \sum_{l \neq 0, \nu}^{n} |N_l| - |N_\nu| \tag{4.9}$$

The number of quanta in the secondary waves hence exceeds their number in the incident wave; i.e., generation of new quanta occurs. This latter can be treated, to a known degree, as the induced Cerenkov radiation of a moving inhomogeneous medium which originates independently of whether the medium itself or the parameter wave moves.

The relationships (4.8) and (4.9) have been obtained earlier in [2] for the particular case of a moving inhomogeneous plasma.

To a definite degree (4.2) and (4.7) are analogous to the known Manley–Rowe relationships obtained earlier [6] for continuous signals in parametric systems; in particular, they permit finding the energy characteristics of wave packets in systems of type (2.1) if their frequency characteristics are known.

5. Let us henceforth examine some particular cases. As a first illustration let us consider plane electromagnetic wave propagation in a fixed, nondispersive dielectric with the traveling parameters ($\varepsilon = \varepsilon(\xi)$, $\mu = \mu(\xi)$). In the one-dimensional case, the system is described by the Lagrangean

$$L = \frac{1}{8\pi}(\varepsilon c_*^{-2} A_t^2 - \mu^{-1} A_x^2) \tag{5.1}$$

where c_* is the speed of light in a vacuum. Here the potential A is selected as the generalized coordinate; the electrical field intensity E and induction B are expressed as

$$E = -c_*^{-1} A_t, \qquad B = A_x$$

Comparing (5.1) and (2.1) we find that here

$$a = -(8\pi\mu)^{-1}, \quad b = 0, \quad c = \frac{\varepsilon}{8\pi} c_*^{-2}, \quad d = 0$$

In conformity with (2.7) the auxiliary problem is described by a Lagrangean of the form (2.1) with the coefficients

$$a' = -\frac{1-\beta^2}{8\pi\mu}, \quad b' = 0, \quad c' = \frac{\varepsilon c_*^{-2}}{8\pi(1-\beta^2)}, \quad d' = 0$$

where $\beta = |V| c_*^{-1} \sqrt{\varepsilon\mu}$; physically this corresponds to electromagnetic wave propagation in a stationary dielectric with the permittivity

$$\varepsilon' = \varepsilon(1-\beta^2)^{-1}, \qquad \mu' = \mu(1-\beta^2)^{-1} \tag{5.2}$$

Let there be just one wave in the initial system (5.1) which does not satisfy the radiation conditions and is incident toward the layer. Then in front of and behind the inhomogeneity, where the parameters ε and μ are constant and equal to ε_1, μ_1 and ε_2, μ_2, respectively, the solution can be written as two waves (let us designate them with the indices, where the lower sign corresponds to a wave moving toward the inhomogeneity), whose frequencies and wave numbers equal in conformity with (3.4)

$$\omega_\pm = \omega'(1 \mp \beta)^{-1}, \qquad k_\pm = k'(1 \pm \beta) \tag{5.3}$$

The layer reflection and transmission coefficients (R and T) for motions at below-light speeds ($\beta_{1,2} < 1$) are found by the substitution of (5.3) into (3.7),

$$R = R' \left(\frac{1+\beta_1}{1-\beta_1}\right)^2, \qquad T = T' \left(\frac{1+\beta_1}{1+\beta_2}\right)^2 \tag{5.4}$$

Let us note that only the condition $\beta_{1,2} < 1$ is required for the validity of (5.4); here there may be $\beta > 1$ in the most inhomogeneous domain. As is seen from (5.4), if the dielectric permittivities are identical on both sides of the inhomogeneous layer ($\beta_1 = \beta_2$); then $T = T'$; however, $R \neq R'$.

For motions "faster than light" the domains of the variable parameters $\beta_{1,2} > 1$ follow from (5.2):

$$\varepsilon' < 0, \quad \mu' < 0$$

Wave propagation in media with negative ε and μ generally has some singularities [7], which do not, however, hinder the application of the formal method elucidated in Sec. 2-4. In this case, no reflected wave originates, and there will be two transmitted waves behind the moving layer. Their transformation coefficients (T_+ and T_-) can be obtained from (5.3) and (3.7) as

$$T_+ = \frac{R'}{T'} \left(\frac{1+\beta_1}{1-\beta_2}\right)^2, \quad T = \left(1 + \frac{R'}{T'}\right)\left(\frac{1+\beta_1}{1+\beta_2}\right)^2 \tag{5.5}$$

For the case of a sharp boundary, R' and T' are easily expressed in terms of the values of the wave resistances on different sides of the jump

$$R' = \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right)^2, \qquad T' = \frac{4\rho_1\rho_2}{(\rho_1 + \rho_2)^2} \qquad (\rho' = \sqrt{\mu'/\varepsilon'} = \sqrt{\mu/\varepsilon} = \rho)$$

hence, the formulas (5.4), (5.5) agree with the results presented in [3].

The case when the quantity $1-\beta_2$ changes sign within the layer is more complex. In particular, for $\beta_1 < 1 < \beta_2$ not two, as is customary, but three secondary waves originate here. Such cases are not examined herein, but they have been investigated in [3] in application to a sharp interface.

The relationships (5.3)-(5.5) somewhat recall the relativistic conversion formulas; however, the method under consideration is more convenient since the material equations in the system K' retain their form, while they are replaced by the more complex Minkowski relations upon going over to the reference system accompanying the parameter wave; moreover, according to (3.3) the wave amplitudes in K and K' are identically equal.

6. The reflection and refraction of a weak acoustic wave by a moving inhomogeneity of parameters of the medium can be examined analogously. The equations of motion for such a system have the same form in the Lagrangean variables η and t [8] as in the previous case, and the Lagrangean can be written as follows:

$$L = \frac{1}{2\rho_0} \varphi_\eta^2 - \frac{\rho_0}{2\rho c_s^2} \varphi_t^2 \tag{6.1}$$

Here $\rho(\xi)$ is the variable density of the medium, $\rho_0 = \text{const}$ is the initial density, $c_s(\xi)$ is the speed of sound in the medium, φ is the velocity potential defined according to the relationships

$$p_s = -\varphi_t, \quad v_s = \rho_0^{-1}\varphi_\eta$$

where p_s and v_s are small particle pressure and velocity perturbations corresponding to a weak signal. It is easy to see that (6.1) agrees with (5.1) to the accuracy of the substitution

$$\rho_0 \to -4\pi\mu, \quad \sqrt{\rho}\, c_s \to 4\pi \sqrt{\mu/\varepsilon}\, c_*, \quad \varphi \to A, \quad \eta \to x$$

so that the deductions in Sec. 5 can be applied directly to this case also.

Using the results of [9] it is easy to show that a Lagrangean of the form (6.1) also describes the one-dimensional problem about plane longitudinal and transverse wave interaction in an elastic isotropic medium.

7. As an illustration of a dispersive system, let us consider plane electromagnetic waves in an inhomogeneous plasma moving at the velocity V in a dielectric with constant parameters ($\varepsilon, \mu = \text{const}$). The term $\frac{1}{2}Nmv^2 + c_*^{-1}jA$, which takes account of the kinetic energy of the electrons and their interaction with

the field, must be added to the expression for the Lagrangean (5.1) in this case; here $j = eNv$ is the density of the currents induced in the plasma, v is the electron velocity perturbation of the electromagnetic wave ($v \ll c_*$), N is the concentration, and e and m the charge and mass of the electrons. Taking into account that in conformity with the equation of electron motion

$$\dot{v} = A - \frac{e}{mc_*} A$$

we obtain

$$L = \frac{1}{8\pi} (\varepsilon c_*^{-2} A_t^2 - \mu^{-1} A_x^2 - \varepsilon c_*^{-2} \omega_p (\zeta) A^2) \quad \left(\omega_p = \sqrt{\frac{4\pi e^2 N}{\varepsilon m}}\right) \tag{7.1}$$

where ω_p is the plasma (Langmuir) frequency. It follows for the system (7.1) from (2.7) that an inhomogeneous plasma $\omega_p'(x) = \omega_p \sqrt{1-\beta^2}$ in a medium with the refraction coefficient

$$n' = \sqrt{\varepsilon' \mu'} = \sqrt{\varepsilon \mu} (1 - \beta^2)^{-1}$$

corresponds to the auxiliary problem in this case.

A similar system has been examined earlier in [2]. The fundamental results of this paper are easily obtained from (3.4), (3.7), and (4.7), where in conformity with (7.1) it is sufficient to put

$$a = -(8\pi\mu)^{-1}, \quad b = 0, \quad c = \frac{\varepsilon}{8\pi c_*^2}, \quad d = -\frac{\varepsilon \omega_p^2}{8\pi c_*^2}$$

The presence of dispersion specifies the appearance of some singularities for wave propagation in systems of the type (7.1). In particular, it turns out that for one incident wave, more than two secondary waves cannot already originate here. (For $\beta < 1$ these are the reflected and refracted waves, and for $\beta > 1$ they are two transmitted waves behind the moving inhomogeneity.) The plasma nature of the dispersion also results in the fact that the transmission coefficient is $T' = 0$ (meaning also $T = 0$) for $\beta < 1$ while $R' = 1$ (a "moving mirror") for the frequencies $\omega' < \omega_{p2} \sqrt{1-\beta^2}$ (where ω_{p2} is the plasma frequency behind the inhomogeneous domain).

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